MTH 201
Multivariable calculus and differential equations
Homework 5
Mean value theorem, maxima and minima

1. Compute the gradient vector and Hessian matrix for each of the following functions
   (a) $f(x, y) = e^{xy} \sin(x^2 + y^2)$
   (b) $f(x, y) = \frac{\sin x}{1+y^2}$
   (c) $f(x, y) = x^2 - y^2$
   (d) $f(x, y) = (x + y)^2 - x^4$.

2. Find an equation of tangent plane to the surface $3xy + z^2 = 4$ at $(1, 1, 1)$.

3. Find unit normal to the ellipsoid $x^2 + 2y^2 + 3z^2 = 10$ at $(\sqrt{10}, 0, 0)$.

4. Check if the following functions satisfy the hypothesis of mean value theorem
   (a) $f(x, y) = x \sin y$
   (b) $f(x, y) = x^2 + y^2$
   (c) $f(x, y) = \sqrt{x^2 + y^2}$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x = 0$ and $f_y = 0$. Then show that f is a constant function.

6. Let $f : B(\mathbf{0}, 5) \to \mathbb{R}$ be a continuous function such that $f(1, 0) = 2$ and $f(0, 1) = 10$. Show that there exists a point $(x_0, y_0) \in B(\mathbf{0}, 5)$ such that $f(x_0, y_0) = 5$. Here $B(\mathbf{0}, 5)$ denotes the open disc in $\mathbb{R}^2$ with radius 5 and centred at $(0, 0)$.

7. Find the critical points of functions given below and determine whether they are local maxima, local minima, or saddle points
   (a) $f(x, y) = (x^2 - y^2)e^{\frac{-x^2-y^2}{2}}$
   (b) $f(x, y) = xy + y^2$
   (c) $f(x, y) = x^4 + y^2$
   (d) $f(x, y) = x^3 + y^3 - 3xy + 4$
   (e) $f(x, y) = (x + y)^2 - x^4$.

8. Find absolute maximum and minimum values of $f(x, y) = x^2+2y^2$ on the disc $x^2+y^2 \leq 1$.

9. Find absolute maximum and minimum values of $f(x, y) = 2x^2 - y^2 + 6y$ on the disc $x^2 + y^2 \leq 16$.

10. Find absolute maximum and minimum values of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular region in first quadrant bounded by $x = 0, y = 0$, and $y = 9 - x$.

11. Find absolute maximum and minimum values of $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the square given by $-1 \leq x \leq 1, -1 \leq y \leq 1$.

12. Find maximum and minimum values of $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$.

13. Find maximum and minimum values of $f(x,y) = x^2 - y^2$ subject to the constraint $y - x = 1$.

14. Find maximum and minimum values of $f(x,y) = 5x - 3y$ subject to the constraint $x^2 + y^2 = 136$.

15. Find maximum and minimum values of $f(x,y,z) = x + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

16. Find maximum and minimum values of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 1$.

17. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

18. Find the shortest distance from the point $(-2, -1, 5)$ to the plane $4x - 2y + z = 1$.

19. Among all triangles with a fixed perimeter find the one with maximum area.